Silting Theory and Derived Base Change

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Derived categories and Cohen–Macaulay representation theory

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Silting theory

Let A be a dg algebra. $M \in \mathrm{per}(A) := \mathrm{thick}_{\mathcal{D}(A)} A$ is a *silting object* if

$$\mathrm{Hom}_{\mathcal{D}(A)}(M, \Sigma^{>0} M) = 0, \qquad \mathrm{thick}\, M = \mathrm{per}(A).$$

Some basic properties

- [Aihara–Iyama12] Silting objects form a poset and admit mutations.
- [Koenig–Yang14] If $A = \Lambda$ is a finite-dimensional algebra, then there is a bijection between
  - basic silting objects of Λ,
  - algebraic t-structures on $\mathcal{D}^b(\mathrm{mod}\ \Lambda)$,
  - bounded weight structures on $\mathcal{K}^b(\mathrm{proj}\ \Lambda)$.

Base reduction problem

- $(R, \mathfrak{m}, k)$: commutative complete local noetherian ring.
- Λ: module-finite R-algebra.

Previous results

- [Kimura24] $I \subseteq \mathfrak{m}\Lambda$: an ideal $\Rightarrow$ $2\text{-}\operatorname{silt}(\Lambda) \simeq 2\text{-}\operatorname{silt}(\Lambda/I)$.
- [Gnedin22] $\mathfrak{a} \subseteq \mathfrak{m}$: ideal s.t. $\operatorname{Tor}^R_{>0}(\Lambda, \mathfrak{a}^n/\mathfrak{a}^{n+1}) = 0$ $(n \geq 0) \Rightarrow$ $\operatorname{silt}(\Lambda) \simeq \operatorname{silt}(\Lambda/\mathfrak{a}\Lambda)$

Closed-fiber reduction theorem [F]

$- \otimes^{\mathbf{L}}_R k \colon \operatorname{silt}(\Lambda) \to \operatorname{silt}(\Lambda \otimes^{\mathbf{L}}_R k)$ is a poset isomrophism.

Closed-fiber reduction theorem recovers previous results.

Ordinary fiber and derived fiber

$$R = k[[x]], \qquad Q:\ 2 \xleftarrow{\alpha} 1 \xrightarrow{\beta} 3, \qquad \Lambda = RQ/(x\alpha, x\beta).$$

Ordinary fiber

$$\Lambda \otimes_R k \simeq kQ \text{ is silting-discrete.}$$

Derived fiber

$$(R\widetilde{Q}, d) \xrightarrow{\simeq} \Lambda, \qquad |\alpha'| = |\beta'| = -1, \qquad d(\alpha') = x\alpha, \quad d(\beta') = x\beta.$$

$$\begin{tikzcd} 2 & 1 \arrow[l, shift right, "\alpha"'] \arrow[l, shift left, "\alpha'"] \arrow[r, shift left, "\beta"] \arrow[r, shift right, "\beta'"'] & 3 \end{tikzcd}$$

$$\Lambda \otimes^{\mathbf{L}}_R k \simeq (R\widetilde{Q}, d) \otimes_R k \simeq (k\widetilde{Q}, 0).$$

$(k\widetilde{Q}, 0)$ is silting-indiscrete [F26] $\Rightarrow$ Λ is silting-indiscrete.

Summary

Closed-fiber reduction theorem [F]

- $(R, \mathfrak{m}, k)$: commutative complete local noetherian ring.
- Λ: module-finite R-algebra.

Then

$$\otimes^{\mathbf{L}}_R k \colon \operatorname{silt}(\Lambda) \to \operatorname{silt}(\Lambda \otimes^{\mathbf{L}}_R k)$$

is bijection.

Next, we generalize the above theorem to the *dg setting*.

General setting

Terminology

connective : $H^i(-) = 0 \ (i > 0)$, *commutative* dg ring : $xy = (-1)^{|x||y|}yx$.

Standing assumptions

- R: connective commutative dg ring s.t.

$$(H^0(R), \mathfrak{m}, k)\text{: complete local noetherian,} \qquad H^i(R) \in \operatorname{mod} H^0(R) \qquad \forall i \in \mathbb{Z}.$$

- A: connective dg R-algebra s.t.

$$H^i(A) \in \operatorname{mod} H^0(R) \qquad \forall i \in \mathbb{Z}.$$

Example

$$R = k[t], \quad |t| = -2, \quad d = 0, \qquad R = k[t]/(t^2), \quad |t| = -1, \quad d = 0.$$

Main theorem: reduction to the derived closed-fiber

Closed-fiber reduction theorem [F]

$$- \otimes^{\mathbf{L}}_R k : \operatorname{silt}(A) \xrightarrow{\sim} \operatorname{silt}(A_k), \qquad A_k = A \otimes^{\mathbf{L}}_R k.$$

Well-definedness

Let $M \in \operatorname{silt}(A)$. Since $M \in \operatorname{thick} A$, we have

$$\operatorname{RHom}_A(M, N) \otimes^{\mathbf{L}}_R k \simeq \operatorname{RHom}_{A_k}(M_k, N_k) \qquad \forall N \in \mathcal{D}(A).$$

M presilting $\Leftrightarrow \operatorname{REnd}_A(M) \in \mathcal{D}(R)^{\leq 0} \Leftrightarrow \operatorname{REnd}_{A_k}(M_k) \in \mathcal{D}(k)^{\leq 0} \Leftrightarrow M_k$ presilting.
Moreover,

$$A \in \operatorname{thick}(M) \implies A_k \in \operatorname{thick}(M_k),$$

so M_k is silting.

Example

Let

$$R = k[t], \qquad |t| = -2,$$

and let Q be the graded 2-cycle

$$1 \underset{b}{\overset{a}{\rightleftarrows}} 2, \qquad |a| = 0, \quad |b| = -2.$$

Put $A = kQ$ and let

$$t \longmapsto ab + ba.$$

Then

$$A \otimes^{\mathbf{L}}_R k \simeq kQ/(ab, ba),$$

and we have

$$\operatorname{silt}(kQ) \cong \operatorname{silt}\bigl(kQ/(ab, ba)\bigr).$$

Main ingredient

Silting–SMC correspondence [F]

There is a bijection $\mathsf{silt}(A) \simeq \mathsf{SMC}(A)$.

A *simple-minded collection* (*SMC*) in $\mathcal{D}^b_{\mathrm{fl}}(A) := \{X \in \mathcal{D}(A) \mid H^*(X) \in \mathsf{fl}\, H^0(R)\}$ is a finite set $\mathcal{L} = \{X_1, \dots, X_n\}$ s.t.

- $\mathrm{Hom}(X_i, \Sigma^p X_j) = 0$ for $p < 0$;
- $\mathrm{Hom}(X_i, X_j) = 0$ for $i \neq j$, and $\mathrm{End}(X_i)$ is a division ring;
- $\mathsf{thick}(\mathcal{L}) = \mathcal{D}^b_{\mathrm{fl}}(A)$.

For $R = k$ field, this type of correspondence was known [Keller–Vossieck88; Keller–Nicolás11; Koenig–Yang14; Su–Yang19; F27]

Sketch of proof

Key lemma [F]

$$\iota : \mathcal{D}^b_{\mathrm{fl}}(A_k) \longrightarrow \mathcal{D}^b_{\mathrm{fl}}(A), \qquad A_k = A \otimes^{\mathbf{L}}_R k,$$

sends SMC to SMC:

$$\mathcal{L} \in \mathsf{SMC}(A_k) \quad \Longrightarrow \quad \iota(\mathcal{L}) \in \mathsf{SMC}(A).$$

The main theorem follows from the commutativity of the following diagram:

$$\begin{tikzcd}
\mathsf{silt}(A) \arrow[r, "- \otimes^{\mathbf{L}}_R k"] \arrow[d, leftrightarrow] & \mathsf{silt}(A_k) \arrow[d, leftrightarrow] \\
\mathsf{SMC}(A) & \mathsf{SMC}(A_k) \arrow[l, "\iota"]
\end{tikzcd}$$

Base change theorem

$R \to S$: local morphism with the same residue field k.

General base change theorem [F]

$$- \otimes^{\mathbf{L}}_R S : \ \mathrm{silt}(A) \xrightarrow{\sim} \mathrm{silt}(A \otimes^{\mathbf{L}}_R S).$$

$$\begin{tikzcd}
\mathrm{silt}(A) \arrow[r, "- \otimes^{\mathbf{L}}_R S"] \arrow[d, "- \otimes^{\mathbf{L}}_R k"'] & \mathrm{silt}(A \otimes^{\mathbf{L}}_R S) \arrow[d, "- \otimes^{\mathbf{L}}_S k"] \\
\mathrm{silt}(A_k) \arrow[r, equal] & \mathrm{silt}((A_S)_k)
\end{tikzcd}$$

Summary

Closed-fiber reduction theorem

$$\mathrm{silt}(\mathrm{per}\, A) \xrightarrow{\sim} \mathrm{silt}(\mathrm{per}\, A_k), \qquad A_k = A \otimes^{\mathbf{L}}_R k.$$

The key tool is the silting–SMC correspondence, which allows us to avoid a direct lifting argument.

References

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